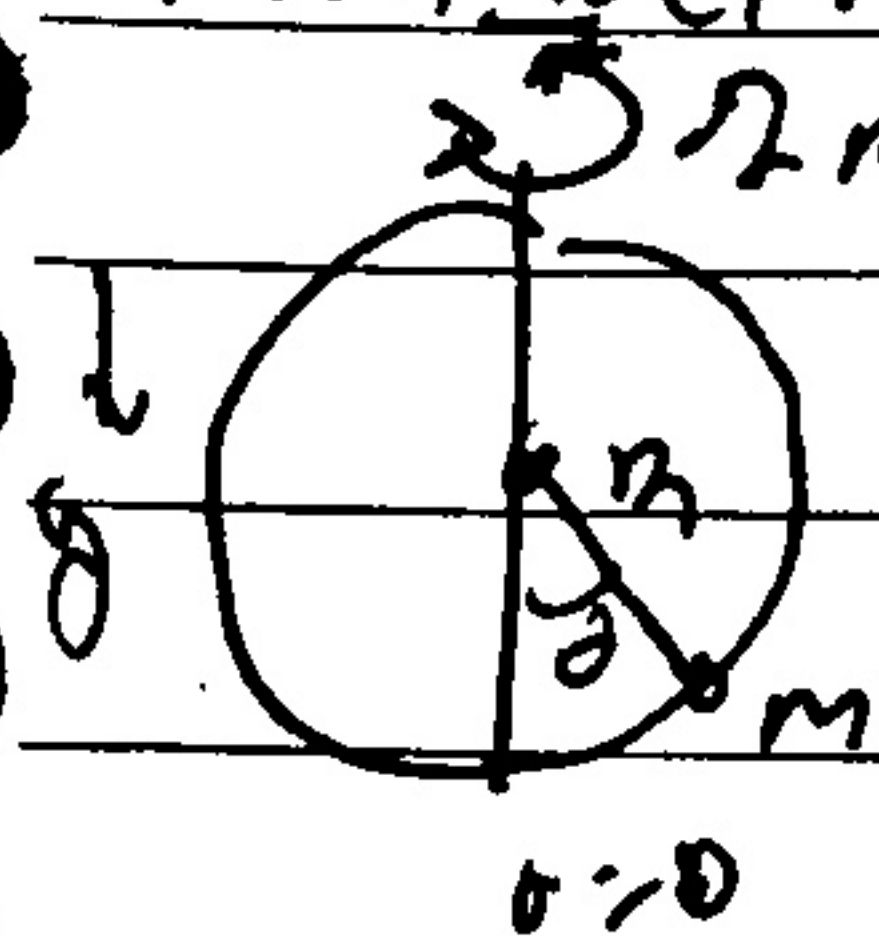


[10/11/2026]

Math Seminar Lecture Notes - Taddeus Kiker

Goal: (1) understand stationary action and (2) why Lagrangian Hamilton's equations describe the same motion when momentum/velocity is invertible.

PART I: EXAMPLE (MOTIVATION)



Q: @ rest, head like the bottom. if we spin the sphere fast enough, is the bottom still still (static friction)?

Need for energy. m & r are $\perp$, sliding $R\dot{\theta}$ and carried $R\dot{\phi}\sin\theta$. Height above bottom is $R(1-\cos\theta)$. Hence, $T = \frac{1}{2}mR^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$, $U = mgR(1-\cos\theta)$

REVIEW: from $L = T - U$ for this ideal system, I will derive the EOM from stationary action and justify next.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}, \quad \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mR^2\dot{\phi}^2\sin\theta\cos\theta - mgR\sin\theta$$

$$\text{Hence } \ddot{\theta} = \sin\theta (R^2\dot{\phi}^2\cos\theta - g/R).$$

Near θ , $\ddot{\theta} \approx (R^2\dot{\phi}^2 - g/R)\theta$. Below $\dot{\phi} = \sqrt{g/R}$ no θ is restored, above it, no $\dot{\phi}$ grows.

$$\text{Stall points: } \cos\theta = g/(R\dot{\phi}^2).$$

Why Lagit?

PART II: FROM PATH TO AN EQUATION

$q(t)$ is path parameterization. Lagrangian $L(q, \dot{q}, t)$, $T - U$, eval @ each instant + integrating it gives the action $S[q]$ on a curve for whole path.

Principle of Least Action: extremizes extrinsic history of the action integral over the (S) or minimum of action. Needs the stationary (for q first)

$q(t) \in \mathbb{R}^n$, with fixed end point times + positions.

$S[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t) dt, q(t_0) = q_0, q(t_1) = q_1$

Does variation exist?

Choose $C = T - U$.

All absolute minima are stationary, not every stationary path is an absolute minimum (nature dictates local behavior).

now we vary this point:

$\delta M(t) = \text{shorter length}$
 $\epsilon = \text{arbitrary size}$

$q_\epsilon = q + \epsilon \eta, \eta(t_0) = \eta(t_1) = 0.$

$\delta S[q; \eta] = \frac{d}{d\epsilon} S[q_\epsilon] \Big|_0$

Stationary means $\delta S = 0$ for every admissible η .
 (eq. action the stationary)

changing path changes position by $\epsilon \eta$ and velocity by $\epsilon \dot{\eta}$.
 structure is $\downarrow$
 path + vel. $\downarrow$ path + vel.

$\delta S = \int_{t_0}^{t_1} (L_q \cdot \eta + L_v \cdot \dot{\eta}) dt$

By parts to isolate the velocity and position derivative from $\dot{\eta}$ to L_v .

$= [L_v \cdot \eta]_{t_0}^{t_1} + \int_{t_0}^{t_1} (L_q - \frac{d}{dt} L_v) \cdot \eta dt.$

End point calculation: if $\int_{t_0}^{t_1} (L_q - \frac{d}{dt} L_v) \cdot \eta dt = 0$ for all η , then $L_q - \frac{d}{dt} L_v = 0$ (anywhere).
 End point terms vanish because we said $\eta(t_0) = \eta(t_1) = 0$.
 here L_q , since the remaining integral must vanish for every η , its coefficient must be zero.

Thus, left w/ Euler Lagrange eq: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

Small change in path changes vel, dir, and pos. Statically requires 1st order change in the integrand of the action. In path B, kinetic energy is higher, while potential energy is lower. Balance = Newton's 2nd law.

Does variational mechanics w/ it derived describe physical nature?

Choose $L = T - U$. For constant masses and a pos dep. potential, $L = \sum_a \frac{1}{2} m_a |\dot{\vec{r}}_a|^2 - U(\vec{r}_1, \dots, \vec{r}_n)$.

$$\frac{\partial L}{\partial \dot{\vec{r}}_a} = m \dot{\vec{r}}_a, \quad \frac{\partial L}{\partial \vec{r}_a} = -\nabla_a U.$$

$$\frac{\partial L}{\partial \dot{\vec{r}}} = m \dot{\vec{r}}, \quad \frac{\partial L}{\partial \vec{r}} = -\nabla U \quad (1D)$$

Momentum change of momentum equals force.

Euler-Lagrange eq, $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

$$m a \ddot{\vec{r}}_a = -\nabla_a U. \text{ Newton's eq.}$$

Stationary Action and Newton's Law describe the same trajectories. Stationary action organizes the calc. so we can use coordinates suited to constraints.

Extra:

1) Poincaré, $\vec{r} = \vec{r}(q, t)$, $\delta \vec{r} = \vec{r}_q \eta$, $\delta S = -\int (m \ddot{\vec{r}} \cdot \nabla U) \cdot \vec{r}_q \eta dt$.

Newton's law $m \ddot{\vec{r}} = -\nabla U + \vec{N}$, (identical $\vec{N} \cdot \vec{r}_q = 0$).

$\delta S = 0$, Conversely, Stationarity says radial force has no allowed tangent component.

NOTE: $\dot{\vec{r}} = \vec{r}_q \dot{q} + \vec{r}_t$, $\vec{N} \cdot \vec{r}_q = 0$, $\vec{N} \cdot \vec{r}_t$ not zero.

2) Lagrangian is not unique.

3) Rigid body example: with normal vectors that rotate w/ the Lmp, $\vec{e}_1 = (\cos(\Omega t), \sin(\Omega t), 0)$, $\vec{e}_2 = (-\sin(\Omega t), \cos(\Omega t), 0)$, $\vec{e}_3 = (0, 0, 1)$; $\vec{r} = R \sin \theta \vec{e}_1 + R (1 - \cos \theta) \vec{e}_2$.

Q: Can we describe this instead with?

i) $|r|^2 \rightarrow \text{small}$

$$\ddot{\mathbf{r}} = R(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta - \Omega^2 \sin \theta) \vec{e}_1 + 2\Omega R \dot{\theta} \cos \theta \vec{e}_2 + R(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) \vec{e}_2$$

$$\ddot{\theta} = \cos \theta \vec{e}_1 + \sin \theta \vec{e}_2 \quad \text{mit Target.}$$

cancel $\dot{\theta}^2$ terms.

$$\ddot{\mathbf{r}} \cdot \vec{e}_\theta = \ddot{R} - R \ddot{\theta} \sin \theta + \dot{\theta}^2 (\cos \theta) \vec{e}_2$$

$$\ddot{\mathbf{r}} \cdot \vec{e}_\theta = R \ddot{\theta} - \Omega^2 R \sin \theta \cos \theta$$

Contact force to wall (project) and give path-matching

$$\text{so } m R (\ddot{\theta} - \Omega^2 \sin \theta \cos \theta) = -m g \sin \theta$$

Shows Lagrangian

PART IV: Legendre transform

ANSATZ:

take general smooth function $f(v)$ with $f'(v) > 0$

on the interval, define new variable by its slope $p = f'(v)$

each v has unique p , knowing p tells you v . (dual points by S. Legendre)

$$\text{def } v(p) = v, \text{ and relation } g(p) = p v(p) - f(v(p))$$

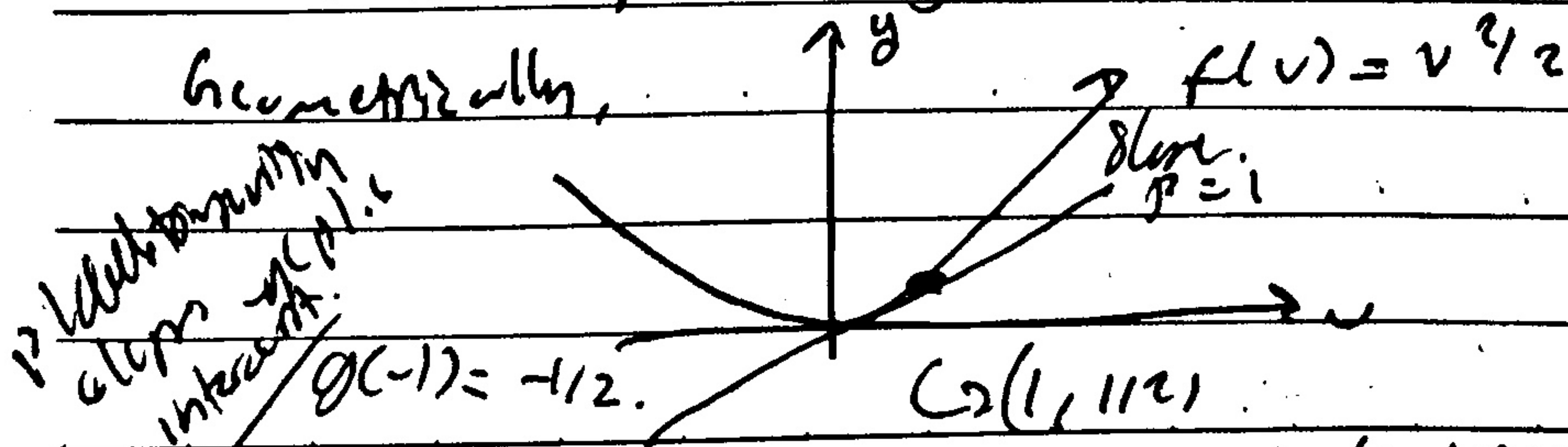
new relation is reverse the derivative relationship

Derive why this can be? differentiating g w.r.t

$$g(p) = v + p \frac{dv}{dp} - \underbrace{f'(v)}_p \frac{dv}{dp} = v$$

extra terms cancel, thus original function has same number of derivatives.

Geometrically,



$$\text{tangent to } f \text{ at } v(p) \text{ is } y = f(v(p)) + p(v - v(p)) = v - g(p)$$

let $\vec{r} = (x, y, z)$

ex: $f(v) = \frac{1}{2} M v^2$, $M > 0$, $p = Mv$, $v = p/M$,
 $g(p) = \frac{p^2}{M} - \frac{p^2}{2M} = \frac{p^2}{2M}$ $\frac{\partial L}{\partial v}$

Applying to mechanics: add q , + fixed, set $p = L(q, v, t)$

Solve for $v(q, p, t)$, and define $H(q, p, t) = p \cdot v(q, p, t) - L(q, v(q, p, t), t)$

Recall $L(q, v, t) = \frac{1}{2} m v^2 - U(q)$, so

$p = \frac{\partial L}{\partial v} = m v$, $v(q, p, t) = p/m$ (rearrange)

replace v in H with $H = p \cdot v - L = H = p \left(\frac{p}{m} \right) - \left(\frac{1}{2} m \left(\frac{p}{m} \right)^2 - U(q) \right)$

Simplified, $H(q, p, t) = \left[\frac{p^2}{2m} + U(q) \right]$

In general, $p_i = \frac{\partial L(q, v, t)}{\partial v_i}$, solve these eqs for velocities $v_i = v_i(q, v, t)$ and substitute into

$H(q, p, t) = \sum_i p_i v_i(q, p, t) - L(q, v(q, p, t), t)$

general. Why? different physical systems - and different coordinates - have different expressions for kinetic and potential energy, so even when $L = T - U$, it need not look like $\frac{1}{2} m v^2 - U(q)$ eg for a rotating wheel $v = \dot{\theta}$, $L = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \dot{\theta}^2 \sin^2 \theta$ (eg L with $\sin^2 \theta$ term)

In summary, this construction is called the Legendre transform of the Lagrangian wrt velocity. We define momentum by $p = \frac{\partial L}{\partial v}$, solve for v in terms of momentum, and form $H = p \cdot v - L$, expressed entirely in q, p, t (a function of momentum, not velocity).

$$f''(x) > 0 \text{ convex}$$

velocity is simply the variable we choose
transforming mech. The general operation
exchanges a variable for the deriv. of function w.r.t
that variable. Here we apply it to $L(q, \dot{q}, t)$ w.r.t
holding q, t fixed.

In general, Legendre transform works for any scalar
function, not just mechanical. For mech $L = K - U$
 $p = f'(x)$. $g(p) = p \cdot x(p) - f(x(p))$. Get
 $x(p)$ by inverting $p = f'(x)$, $g'(p) = x(p)$.
in several variables, $p = \nabla f(x)$, $g(p) = p \cdot x(p) - f(x(p))$.

nondegenerate Hessian makes inversion well defined with
 $L \exp: \text{trail vel w/ momentum}$. Since $p = L_v \dot{q}$
derivative of this map's velocity Hessian

$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$. Note that $W \neq 0$ means
finite then even guaranteed a unique smooth
inverse $v = v(q, p, t)$ near the point considered.

(convex, local). Arnold assumes the stronger condition
that W is ~~velocity~~ positive definite finite (positive
curvature in every velocity direction) which guarantees
nonsingularity

For a bead, $W = mR^2 > 0$, and there is simply $v = \frac{p}{m}$
velocity Hessian: matrix of second derivatives of L w.r.t
velocity variables: $W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$. Since $p_i = \frac{\partial L}{\partial \dot{q}_i}$, it tells you
how momentum changes w/ vel. does - $W_{ij} = \frac{\partial^2 p_i}{\partial \dot{q}_j}$.
For beads only, $\partial \dot{q}_i = \partial_j$, so $W_{ij} = m \delta_{ij}$.

$u + \text{terms. ndep of } v \Rightarrow w = \frac{\partial L}{\partial v} = mR^2.$

I: From Euler Lagrange to Hamilton's Equations.

Goal: show Hamilton's eqs describe exactly the same motion as Euler Lagrange, and check that too lead.

2. differentiate $H = p \cdot v - L$: (use product rule on $p \cdot v$ and chain rule on $L(q, v, t)$)

$$dH = v dp + p dv - L dq - \frac{\partial L}{\partial v} dv - \frac{\partial L}{\partial t} dt.$$

but terms cancel because momentum def as $p = \frac{\partial L}{\partial v}$

$$dH = v dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt.$$

3. because H is function of q, p, t , we also have $dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt$
 matching terms, $\frac{\partial H}{\partial p} = v$, $\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$, $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

3. Bring in Euler Lagrange to obtain Hamilton's eqs.
 for actual trajectory, $H = 0$ with $v = \dot{q}$, $\dot{q} = \frac{\partial H}{\partial p}$.

for 2nd eq, starting w/ definition of momentum? $p = \frac{\partial L}{\partial v}$

(differentiate and apply Euler Lagrange:)

$$\dot{p} = \frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q} \quad \left| \begin{array}{l} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{array} \right.$$

Euler Lagrange 8v!

3. Going backwards: Hamilton implies E-L:

Suppose $q(t), p(t)$ satisfy Hamilton's eqs.

first gives $\dot{q} = v(q, p, t)$ (w/ $\frac{\partial H}{\partial p} = v(q, p, t)$)

Because this $v(q, p, t)$ is inverse of $p = L_v(q, v, t)$,

$p = \frac{\partial L}{\partial v}(q, \dot{q}, t)$. diff and use 2nd Ham eq.

$$\frac{d}{dt} \frac{\partial L}{\partial v} = \dot{p} = -\frac{\partial H}{\partial q} = \frac{\partial L}{\partial q} \quad \left| \frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial q} \right|$$

So the descriptions are equivalent
the vel-momentum relation is invertible

val. for
cylinder. the
really converts

U, Apply to Bead

write $I = mR^2$ choose g as pt. zero @ bottom.

$$L = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} I R^2 \sin^2 \theta - mgR(1 - \cos \theta)$$

↳ find momentum and invert: $\frac{\partial p}{\partial \theta} = \frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} \Rightarrow \dot{\theta} = \frac{\partial p}{\partial I}$

↳ sub into $H = \frac{\partial p}{\partial \theta} \dot{\theta} - L$:

$$H = \frac{\left(\frac{\partial p}{\partial \theta}\right)^2}{I} - \left[\frac{\left(\frac{\partial p}{\partial \theta}\right)^2}{2I} + \frac{1}{2} I R^2 \sin^2 \theta - mgR(1 - \cos \theta) \right]$$

$$S. H = \frac{\left(\frac{\partial p}{\partial \theta}\right)^2}{2I} + mgR(1 - \cos \theta) - \frac{1}{2} I R^2 \sin^2 \theta$$

$$\text{derivs: } \dot{\theta} = \frac{\partial H}{\partial p_{\theta}} = \frac{\partial p}{\partial \theta} \frac{1}{I}, \quad \frac{\partial H}{\partial \theta} = mgR \sin \theta - I R^2 \sin \theta \cos \theta.$$

$$\text{So } \frac{\partial p}{\partial \theta} = \frac{\partial H}{\partial \dot{\theta}} = p_{\theta} = -\frac{\partial H}{\partial \theta} = n(u)$$

$$\text{diff } \dot{\theta} = p_{\theta}/I \quad \text{w/ constant: } \ddot{\theta} = \frac{\dot{p}_{\theta}}{I} = \frac{1}{I} \left(mgR \sin \theta - \frac{1}{2} I R^2 \sin 2\theta \right)$$

$$\text{Hence, } \ddot{\theta} = R^2 \sin \theta \cos \theta - g/R \sin \theta$$

5. Show what is conserved - and distinguish it from lab energy.

Along the trajectory, chain rule gives $\frac{dH}{dt} = H_{\theta} \dot{\theta} + H_{p_{\theta}} \dot{p}_{\theta} + H_t$

In var. H and time: $\frac{dH}{dt} = H_{\theta} \dot{\theta} + H_{p_{\theta}} (-H_{\theta} \dot{\theta}) = 0$

because R, m, g and I has no explicit time dependence

$$\frac{dH}{dt} = 0, \text{ but } H \neq E_{\text{lab}}: T + U = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} I R^2 \dot{\phi}^2 \sin^2 \theta + mgR(1 - \cos \theta)$$

$$\text{me. while sub } p_{\theta} \text{ into } H: H = \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} I R^2 \sin^2 \theta$$

$$S. E_{\text{lab}} = H = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} I R^2 \dot{\phi}^2 \sin^2 \theta + mgR(1 - \cos \theta)$$

(1-290)

Method:

Table

usual, for a regular Lagrangian w/ generalized coordinates, the Legendre transform $(q, \dot{q} \rightarrow (q, p)$,

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

(usually converts 2 second order E-L eqs into

2nd order Hamiltonian eqs: $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

these describe same dynamics w/ same initial

data. Moreover, $\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$, so whenever

H has no explicit time dep, E is conserved.

WRT QM, recall commutator and Heisenberg equation of motion: $\frac{d\hat{A}}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{A}] + \frac{\partial \hat{A}}{\partial t}$,

$$[\hat{H}, \hat{A}] = \hat{H}\hat{A} - \hat{A}\hat{H}$$

Setting $\hat{A} = \hat{H}$, any operator commutes w/ itself, so

$$[\hat{H}, \hat{H}] = 0 \Rightarrow \frac{d\hat{H}}{dt} = \frac{\partial \hat{H}}{\partial t}$$

This motivates QM: the Ham. generates time evolution, but its bracket, quantum mechanically, vanishes. Hence, a Hamiltonian w/ no explicit time dep is conserved.

PART IV: Recap!

$$\delta S[q] = 0 \iff$$

Stationary Action

$$\frac{\partial L}{\partial t} = \frac{\partial L}{\partial q}$$

Euler-Lagrange

Legendre transform
 $p = \frac{\partial L}{\partial \dot{q}}$
 invertible

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$p = -\frac{\partial H}{\partial q}$$

Hamiltonian