Outline.

~10 mins

1. Recap of point set topology.

open sets in $\mathbb{R}$ & continuity → union of balls or every pt contained in a ball.
(intervals)

open sets in $\mathbb{R}^n$ & continuity (homeomorphism)
(balls vs boxes)
& (subsets)

& Smoothness.

open sets in a space. (given by a basis)
&

2. Charts on ~~topological~~ topological.. manifolds.
(Hausdorff, 2nd countable)
↓ "not too tight" ↓ "not too big"

a. constant n.

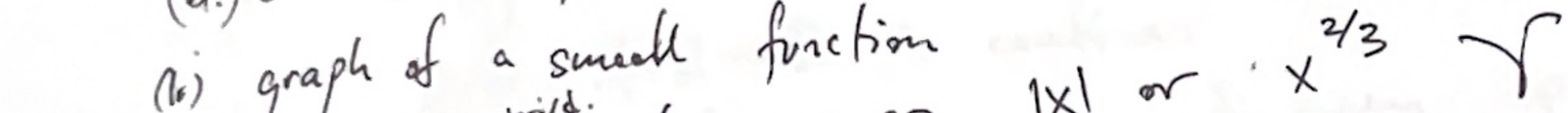
b. local not global.

c. Not unique.

3. Examples

(a.) Euclidean space itself

(b.) graph of a smooth function
→ weird example: cusp $|x|$ or $x^{2/3}$



2) → non example.

3) S^1 three possible charts

i) "cheating" use polar coordinates

ii) locally a graph

iii) stereographic proj

4. smoothness (compatibility & Atlas)

5. Final examples. S^n, $\mathbb{RP}^n$

Topology + Physics Lecture 1 : Felix Roz

Definition + examples of smooth manifolds.

1. Review of point-set topology

a) ~~open~~ $\mathbb{R}$ intervals (a,b) are called <u>open</u>.

any union (arbitrary, finite, countable, uncountable etc.) of open intervals is called an open set.

<u>ex</u>: $\emptyset$, $\mathbb{R}$, $(1,2) \cup (3,4)$, $\mathbb{R} \setminus \{0\}$

<u>equivalently</u>: every pt $x \in \mathcal{U}$ is contained in an open interval $x \in I \subset \mathcal{U}$

continuity: familiar ε-δ def:

$\lim_{x \to a} f(x) = L$: $\forall \varepsilon > 0 \; \exists \delta$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

equivalently: $\forall \varepsilon > 0 \; \exists \delta$ s.t. $x \in (a-\delta, a+\delta) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

or ... $\forall$ open $I \ni L$, $\exists$ open $T \ni a$ s.t. $f^{-1}(I) \supseteq T$

more simply:

<u>Def</u> $f : \mathbb{R} \to \mathbb{R}$ is continuous if the preimage of every open ~~interval~~ set ~~contains~~ is open.

b) $\mathbb{R}^n$ ~~open~~ open "rectangles" $[a,b] \times [c,d] \times \cdots \times [e,f]$

open "balls" $B(r,x) = \{ y \in \mathbb{R}^n \mid \text{dist}(x,y) < r \}$

open set is any union of open rectangles or balls

def <u>homeomorphism</u> continuous w/ cont. inverse.

c) subsets of $\mathbb{R}^n$: open = $\cap$ w/ $A \subset \mathbb{R}^n$ ex.

d) open sets in a space.

② X any set. $\tau \subset P(X)$ "collection of open sets"

1) $X, \emptyset \in \tau$
2) closed under finite intersections
3) arbitrary unions.

2. Charts for Top manifolds (cont.)

Def ~~a top space is locally Euclidean~~

Let M be a top space.

a chart near $x \in M$ is open set $\mathcal{U} \ni x$

together w/ a (homeo morphism) $\phi: \mathcal{U} \to V \subseteq \mathbb{R}^n$
bijection, & continuous. ⟺ ↖ some open in $\mathbb{R}^n$!

(usually we make $\phi(x) = 0$

idea:

Def M is a top manifold if

every pt has a chart { M is Hausdorff & 2nd countable

collection of charts is called an

Rmk 1. n is const. called dim.

2. local, not global property

3. Not unique!

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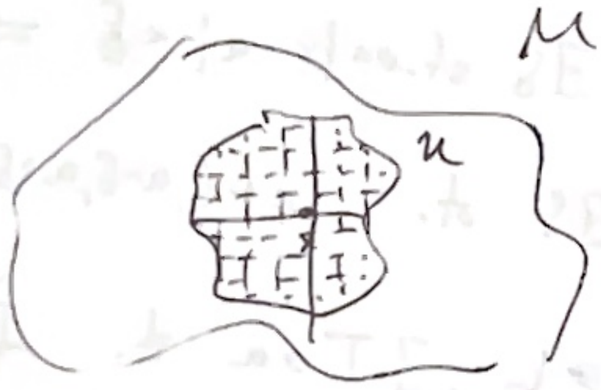


Def M is a top manifold if every pt has a chart

(M is Hausdorff & 2nd countable)

collection of charts is called an atlas

Rmk 1. n is const. called dim.

2. local, not global property

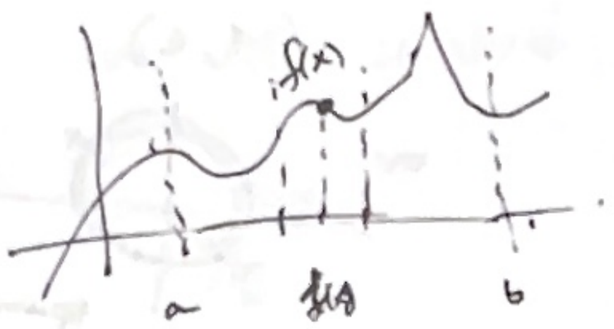
3. Not unique!

③ Examples

a) Euclidean space itself! & open subsets, pfs

pick $x \in \mathbb{R}^4$. choose ~~open~~ open set $\mathbb{R}^4$!

1) graphs of cont. functions.



$f : (a,b) \to \mathbb{R}$.

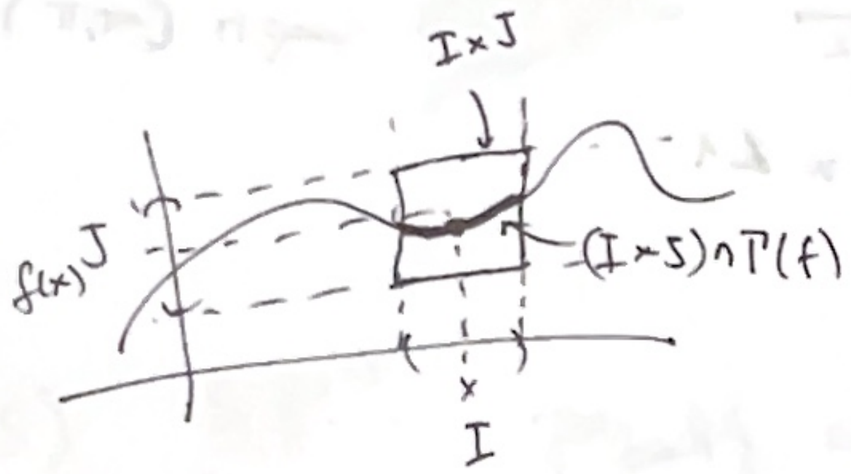
pick $x \in (a,b)$ $f(x) \in \Gamma(f)$

$\{(x, f(x)) \in \mathbb{R}^2\}$

pick I open interval around x.

& J open around $f(x)$

$\mathbb{R}^2$.



$I \times J$ is open in $\mathbb{R}^2$

so $(I \times J) \cap \Gamma(f)$ is open in $\Gamma(f)$

$\phi : (I \times J) \cap \Gamma(f) \longrightarrow I$

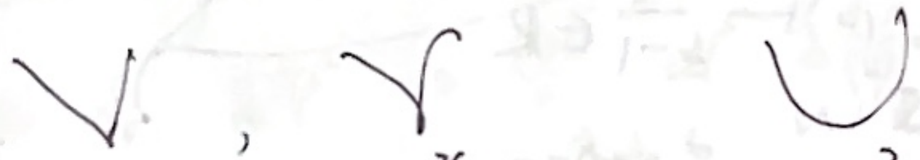
$(x,y) \longmapsto x$ ✓ is continuous

$\phi^{-1} : I \longrightarrow (I \times J) \cap \Gamma(f)$

$x \longmapsto (x, f(x))$ is cont!

so $(\phi, (I \times J) \cap \Gamma(f))$ is a chart for any $f(x) \in \Gamma(f)$

Thus Γ_f is a 1-manifold!

in particular  , ...

2) Non example:

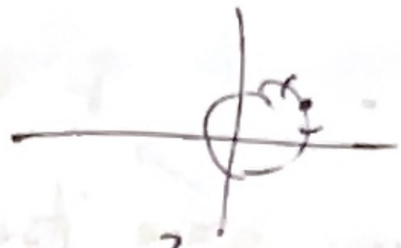
x & y axis.

every open around here ⟷

any map to $\mathbb{R}^1$ will be not injective

map to $\mathbb{R}^2$ or higher w/ not landing in an open set!

(4) 3) $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\} \subseteq \mathbb{R}^2$



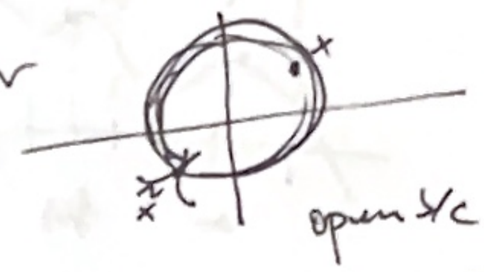
i) use polar coords. $S^1 = \{(r,\theta) \mid r = 1\}$

$x = (1,\theta) \in S^1$ take $(1, \theta+\varepsilon)$ to $(1, \theta-\varepsilon)$

$\downarrow$

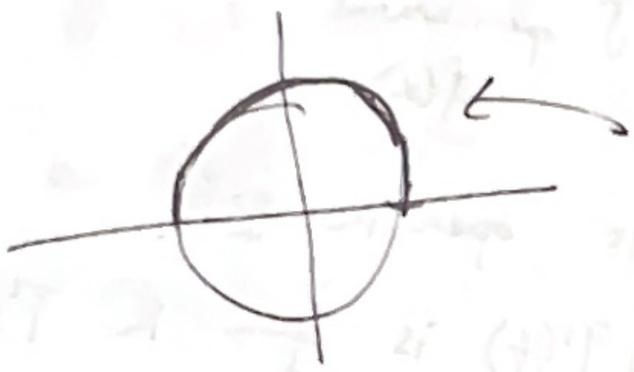
$(-\varepsilon, \varepsilon)$

(use $e^{i\theta}$ if viewed as subset of $\mathbb{C}$) &



open nbhd $S^1 \cap \mathbb{R}^2 \setminus \{\tilde{x}\}$ maps to $(-\pi, \pi)$

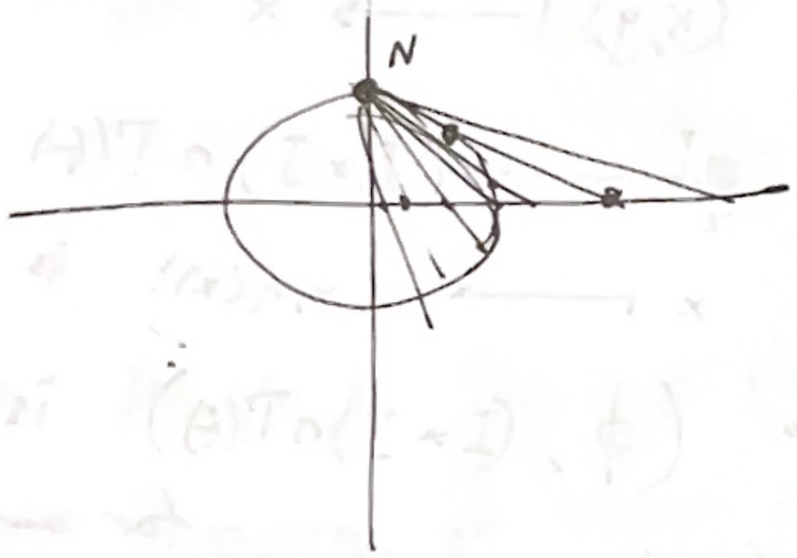
ii) locally a graph



$y = \sqrt{1-x^2}$ for $x < 1$

$\times 4$

iii) Stereographic projection.



start w/ $N = (0,1)$

pt $= (a,b)$

slope of line is $\frac{b-1}{a}$

Line is $y = \frac{b-1}{a}x + 1 \rightarrow$ y-int.

we want pt $(x,0)$ satisfying.

$0 = \frac{b-1}{a}x + 1$

$\frac{-a}{b-1} = x$

Defines map. $(a,b) \mapsto \frac{-a}{b-1} \in \mathbb{R}$

$\uparrow$

$S^1 \setminus N$ & defines $(0,-1) \mapsto 0$

inverse pt $(c,0)$

slope is $\frac{-1}{c}$ $\rightsquigarrow$ line is $y = \frac{-x}{c} + 1$

need solution on $x^2+y^2=1$

$x^2 + \left(\frac{-x}{c}+1\right)^2 = 1$

$= x^2 + \frac{x^2}{c^2} - \frac{2x}{c} + 1 = 1$

soln $x\left(\left(1+\frac{1}{c^2}\right)x - \frac{2}{c}\right) = 0$

soln $x = \frac{2}{c}\left(1+\frac{1}{c^2}\right)^{-1}$

$= \frac{2}{c\left(1+\frac{1}{c^2}\right)}$

now solv for y

⑤ ~~Smooth?~~

4) $S^{n-1} = \{(x_1 \cdots x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^n$

easiest is to use approach ii). but iii) also works

we will construct $2n$ charts ~~one~~ two for each axis.

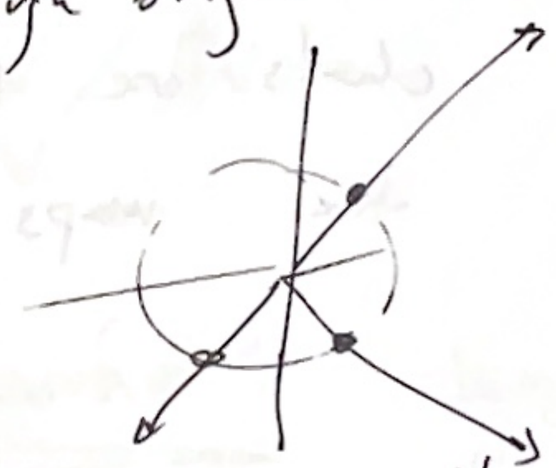
$x_1 = \sqrt{1 - x_2^2 - x_3^2 + \cdots - x_n^2}$

locally a graph.

5) $\mathbb{RP}^n$: probably a new space.

$= \{$ lines through origin in $\mathbb{R}^{n+1}\}$

idea: $S^n/\sim$ is rays through origin in $\mathbb{R}^{n+1}$



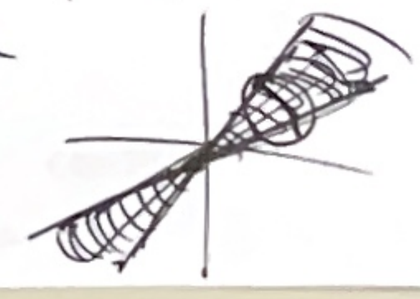
but lines are made of two rays!

so think of $\mathbb{RP}^n$ as S^n w/ opposite pts identified.

more formally. $\mathbb{RP}^n = \mathbb{R}^{n+1} / \mathbb{R}^\times$ scaling

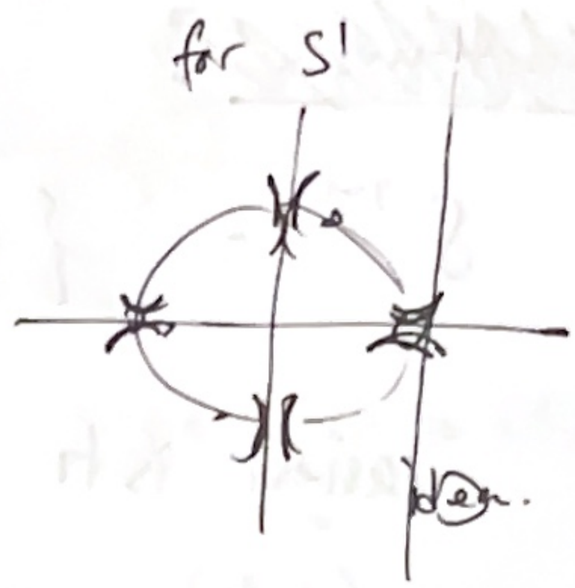
$[a:b] \sim [ka:kb]$ are pts on the same line.

(Note: opens in the quotient if its preimage is open)



(6)

4 charts on $\mathbb{RP}^1$



$$U_0 = \{[a_0 : a_1] \mid a^0 \neq 0\}$$

$$U_1 = \{[a_0 : a_1] \mid a^1 \neq 0\}$$

$\phi_0 : U_0 \longrightarrow \mathbb{R}^1$

$[a_0 : a_1] \sim [1 : \frac{a_1}{a_0}] \longmapsto \frac{a_1}{a_0}$

$\phi_0^{-1} = \mathbb{R}^1 \longrightarrow U_0$

$r \longmapsto [1 : r]$

$\phi_1 : U_1 \longrightarrow \mathbb{R}^1$

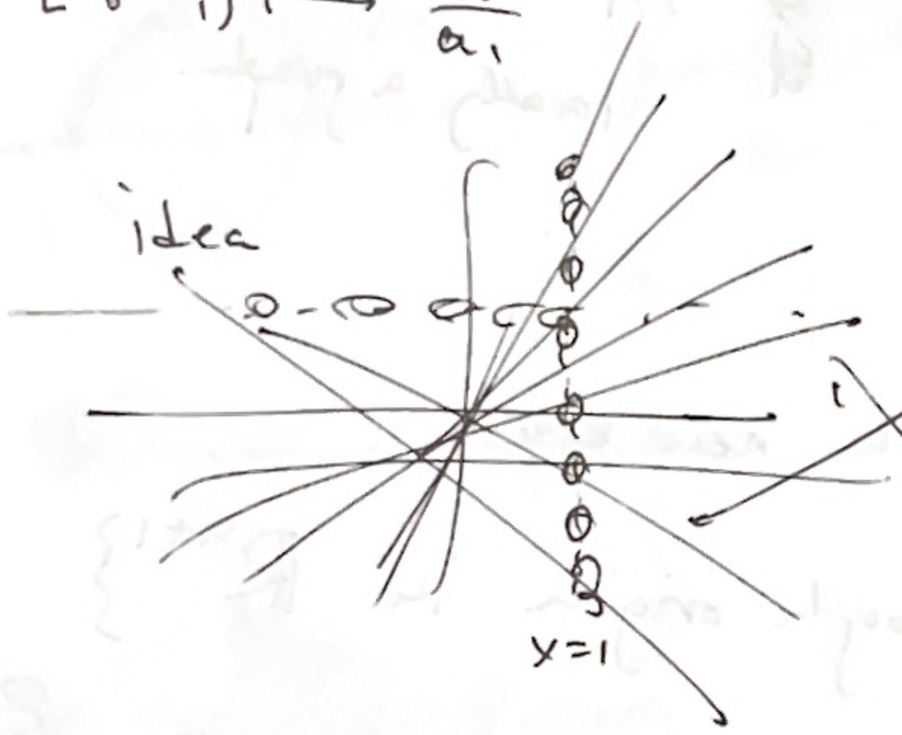
$[a_0 : a_1] \longmapsto \frac{a_0}{a_1}$

$\phi_1^{-1} \quad r \longmapsto [r : 1]$

why well defined?

b/c scaling a_0 also scales a_1!

ratio aka slope is always the same.



gets every line except $X = 0$

every line except $y = 0$

charts are like local graphs

the maps are like stereographic.

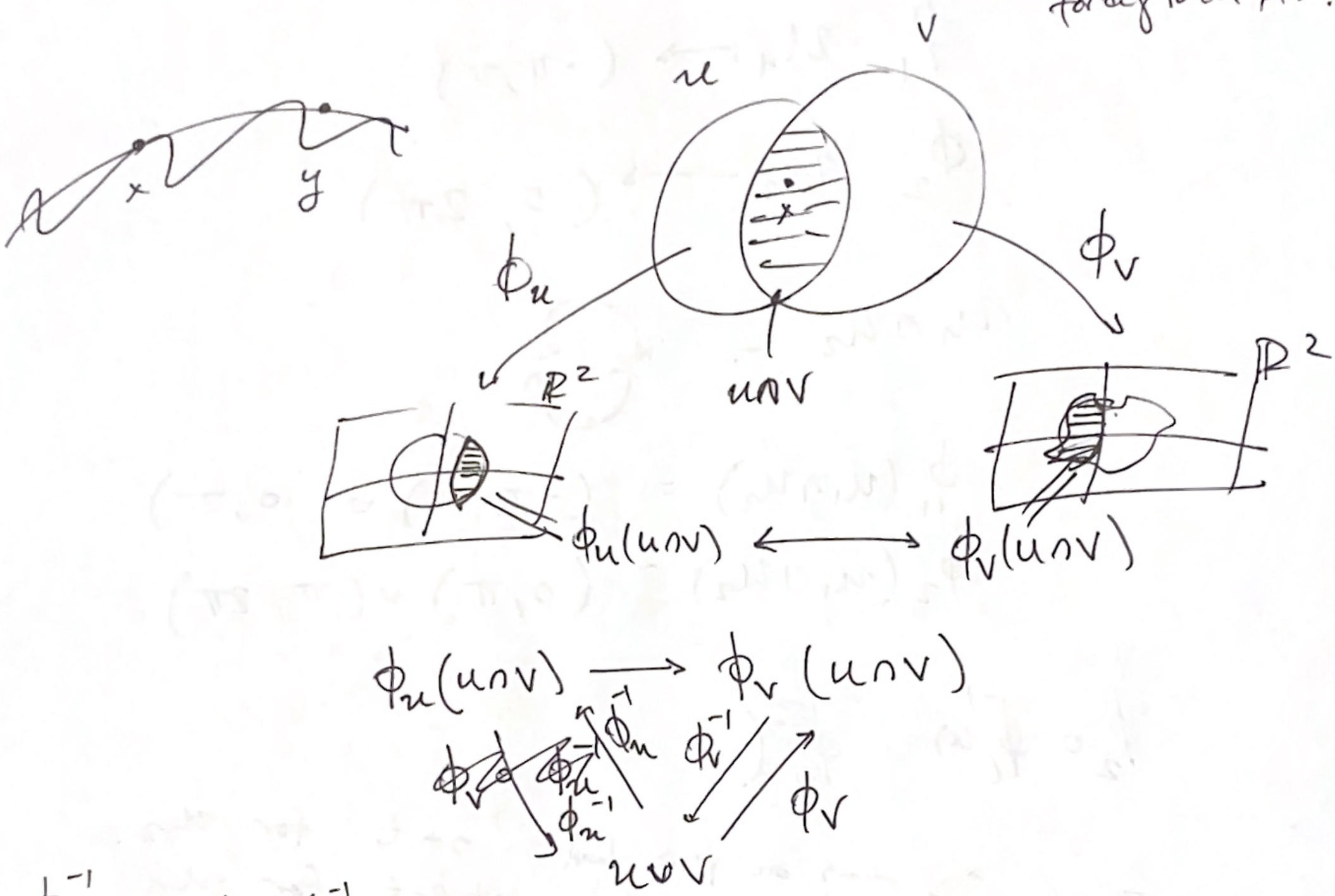
HW: generalize to $\mathbb{RP}^n$.

$U_i = \{[a_0 : \cdots : a_n] \mid a_i = 0\}$.

$\phi_i \quad [a_0 : \cdots : a_n] \longmapsto [a_0/a_i \cdots : 1 : \cdots a_n/a_i]$.

5. Smoothness

basic question is what happens when there are two charts, for a given pt.?



$\phi_V \circ \phi_U^{-1}$ & $\phi_U \circ \phi_V^{-1}$

are called transition functions

they are always defined as functions btwn subsets of $\mathbb{R}^n \to \mathbb{R}^n$.

Def if the transition funcs are C^∞ (smooth) we call this a smoothly compatible atlas

a smooth manifold is a manifold w/ a

maximal smooth atlas

(i.e not contained in a larger atlas) → point is it contains all possible compatible charts.