

Overview:

1) (Ch 1.) Newton & Galileo's Principles + (Ch 1.2E.)

2) later 3) (*) Focus on Lusajai's curves (sec. 5) + Kepler problem (sec. 8)

Start by setting up the machinery for these main examples (sec 4, 6, 7)

2) Talk about phase space as a guiding principle, set up in terms of manifold theory

phase space A^n

We begin by defining "affine space" in $\mathbb{R}^n$.



The picture to think about is that this is a model of $\mathbb{R}^n$ but without any fixed pts. Since the universe is constantly expanding, we take A^4 as a "Galilean" space time structure for the universe.

1) $\mathbb{R}^n$ acts on A^n transitively by: $a \mapsto a+b$, $a \in A^n, b \in \mathbb{R}^n$

represent this as

As for time, we have a linear mapping $T: \mathbb{R}^4 \rightarrow \mathbb{R}$ from the vector space of parallel lines to the real "time axis":

2) The time interval from an event $a \in A^4$ to $b \in A^4$ is given by

$T(b-a)$ [If $T(b-a)=0$, these events are "simultaneous"]

The space of events simultaneous with a given event forms a 3-dim affine subspace in A^4 given by $\ker(T)$.

3) We have some sense of a metric given by: $\rho(a,b) = \|a-b\| = \sqrt{\langle a-b, a-b \rangle}$

$a, b \in A^3$ given by the standard inner product. $\langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$ in $\mathbb{R}^3$
A space A^4

Equipped w/ all of this we have a sense of describing "(non)-simultaneous events in the universe". By providing an abstract way of thinking about how geometric points behave in a space.

A^4 also forms a group under affine transformations

- Galilean group is the group of all affine transformations of A_4 which preserve intervals of time & distance between simultaneous events.

Just as a brief example, consider ~~these~~ affine transformations are called Galilean:
~~uniform motion with velocity $\vec{v}$~~

- A motion in $\mathbb{R}^N$ is essentially our previous notion of an integral curve for a manifold, i.e.

* Def. A differentiable mapping $\gamma: I \rightarrow \mathbb{R}^N$ for $I \subset \mathbb{R}$ where we have a velocity vector at a pt $t_0 \in I$:

$$\dot{\gamma}(t_0) = \left. \frac{d\gamma}{dt} \right|_{t=t_0} = \lim_{h \rightarrow 0} \frac{\gamma(t_0+h) - \gamma(t_0)}{h} \in \mathbb{R}^N$$

Def. $\gamma: I \rightarrow M$ s.t wrt a vector field X , differential is derivative given by:
 $\gamma'(t) = X|_{\gamma(t)}$

& acceleration vector (i.e. 2nd-derivative)

$$\ddot{\gamma}(t_0) = \left. \frac{d^2\gamma}{dt^2} \right|_{t=t_0}$$

Integral curve thm: For $p \in M$, there's an open interval $I_p \subset \mathbb{R}$ & ^{unique} integral curve $I_p \xrightarrow{\phi_p} M$ s.t $\phi_p(0) = p \in M$ & I_p is the maximal interval for such a curve.

§ Newton's Equations

- A motion of a system of n -pts gives, in galilean space, n "world-lines" described by n -motions $\gamma_i: \mathbb{R} \rightarrow \mathbb{R}^3$, $i \in \{1, \dots, n\}$

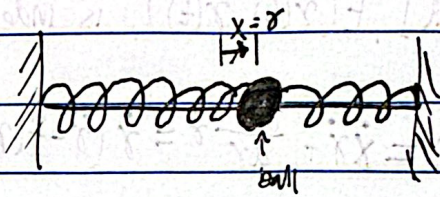
↳ Noting this, according to Newton's principle of determinacy, all motions of a system are uniquely determined by their initial positions & velocities. In other words, there exists a function $F: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$ s.t.

$$(*) \quad \ddot{\gamma} = F(\gamma, \dot{\gamma}, t)$$

Serving as the basis of mechanics, this is Newton's equation

- Galileo's relativity principle $\Rightarrow$ "In physical space-time, there's a selected galilean structure saying if we subject all world lines of a mechanical system, we get world lines of the same system with some new initial conditions."

ex: Consider motion of a weight along a line under the action of a spring:



$$\ddot{r} = -\alpha^2 r \quad (*)$$

Under small extensions of the spring, this is a system whose equation of motion of the weight is given by $(*)$

- ie for,
- 1) open U of $\mathbb{R}^n$
 - 2) open interval I of $\mathbb{R}$
 - 3) smooth map $\Phi: I \times U \rightarrow M$, we have

This eqn can also be written as: $[U = \frac{\alpha^2 r^2}{2}]$ if we introduce the potential energy.

If we replace the one weight we have by two weights, then under the same extension of our spring, the acceleration is half as large.

$$\begin{cases} \frac{\partial \phi(t, q)}{\partial t} = X(\phi(t, q)) \\ \phi(0, q) = q \end{cases}$$

S: Phase spaces + Degrees of Freedom:

Def. A system w/ 1 degree of freedom is described by one differential equation:

$$\ddot{x} = f(x) \text{ for } x \in \mathbb{R}$$

1) Kinetic energy is the quadratic form:

$$T = \frac{1}{2} \dot{x}^2$$

2) Potential energy is given by: $U(x) = - \int_{x_0}^x f(\xi) d\xi$

* Something important to notice is that the potential energy actually determines our f . Therefore, specifying a system of 1-deg of freedom like above is enough to give potential energy.

Def. Total energy is the sum: $E = T + U$

• Law of Conservation (thm): Total energy of pts moving according to our system $[\ddot{r} = f(r), r \in \mathbb{R}]$, $E(r(t), \dot{r}(t))$, is independent of t

↳ pr. $E = \frac{d}{dt}(T+U) = \ddot{r}\dot{r} + \frac{dU}{dr}\dot{r} = \dot{r}(\ddot{r} - f(r)) = 0 \quad \square$

• Our system is equivalent to a system of 2 equations:

$$(*) \quad \begin{cases} \dot{r} = y \\ \dot{y} = f(r) \end{cases}$$

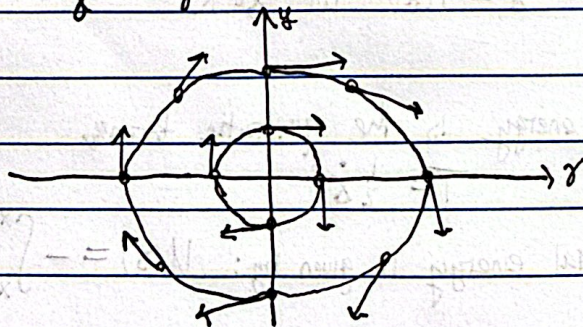
↳ A solution of $(*)$ is an ^{integral curve} $\varphi: \mathbb{R} \rightarrow \mathbb{R}^2$ of a "phase pt" in the "phase plane" [i.e. the plane with coordinates (r, y)] s.t. ^{moving} velocity of a "pt" in time is equal to the phase velocity vector ^{at} the location of the phase pt at that ^{moment} in time

ie: ↳ The image of $\varphi: \mathbb{R} \rightarrow \mathbb{R}^2$ of a phase pt in the phase plane is given by parametric eqns:

$$r = \varphi(t), \quad \dot{y} = \dot{\varphi}(t) = X|_{\varphi(t)} \quad \text{for } X = \text{phase velocity field vector}$$

* exm: The basic eqn for theories of oscillation is:

$$\ddot{x} = -x$$



Here, we have $\begin{cases} T = \dot{x}^2/2, & U = x^2/2, & E = T+U \end{cases}$

• The energy level sets are concentric circles & the origin. The phase velocity vector at the phase pt (x, y) has components $(y, -x)$, where its perpendicular to the radius vector & equal to it in magnitude

$$\frac{1}{2} \dot{y}^2 + U(r) = E$$

(cont.) then, the motion of phase pts in the phase plane are uniform motions about 0, where:

$$r = r_0 \cos(\varphi_0 - t), \quad y = r_0 \sin(\varphi_0 - t)$$

each energy level set is a phase curve.

° each level set is a ~~closed~~ smooth curve in a nbhd of its pts which is not an equilibrium position. $[x = \xi, y = 0]$ is an equilibrium pos. if ξ is a critical pt of potential energy.] $\rightarrow$ b/c of implicit fct thm:

System of 2-deg of Freedom

2-Deg:

For energy level set $\frac{1}{2} \dot{y}^2 + U(r) = E$,

° Def. systems of 2-deg defined by

$$\ddot{x} = f(x) \quad \text{for } x \in E^2$$

where $f = \vec{X}$ is a vector field on the phase plane.

IFT: For C^∞

$f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$,

if $(a, b) \in U$ has

$f(a, b) \neq 0$ &

$\det \left[\frac{\partial f_i}{\partial y_j} \right]_{(a, b)} \neq 0$

then $\exists$ a nbhd

$A \times B \subset U$ of (a, b)

& unique fct

$h: A \rightarrow B$ s.t

$f(x, y) = 0 \Leftrightarrow y = h(x)$

° Thm. Total energy of a conservative system [i.e. $\exists$ a fct $U: E^2 \rightarrow \mathbb{R}$ s.t $f = -\frac{\partial U}{\partial x}$. conservative system's eqn of motion is given by

$\ddot{x} = -\frac{\partial U}{\partial x}$ is conserved s.t

$$\frac{dE}{dt} = 0, \quad \text{for } E = \frac{1}{2} \dot{x}^2 + U(x), \quad \dot{x}^2 = (\dot{x}, \dot{x})$$

° Then, expanding from the phase plane, the phase space is given by the system:

$$\begin{aligned} \dot{x}_1 &= y_1, & \dot{y}_1 &= -\frac{\partial U}{\partial x_1} \\ \dot{x}_2 &= y_2, & \dot{y}_2 &= -\frac{\partial U}{\partial x_2} \end{aligned}$$

Lissajous figures:

$$\ddot{x}_1 = -x_1, \quad \ddot{x}_2 = -\omega^2 x_2$$

$$\text{potential energy} = U = \frac{1}{2} x_1^2 + \frac{1}{2} \omega^2 x_2^2$$

Law of conservation of energy $\Rightarrow$ At initial moment of time, total energy is:

$$\frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) + U(x_1, x_2) = E$$

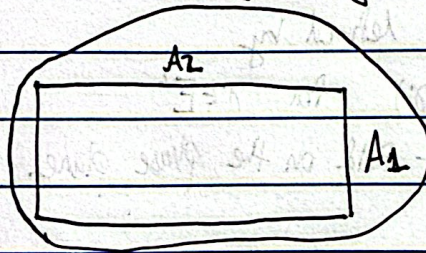
Then, all motions take place in an ellipse: $U(x_1, x_2) \leq E$

This forms two independent 1D systems^{each} satisfying the law of conservation of energy:

$$E_1 = \frac{1}{2} \dot{\gamma}_1^2 + \frac{1}{2} \gamma_1^2, \quad E_2 = \frac{1}{2} \dot{\gamma}_2^2 + \frac{1}{2} \omega^2 \gamma_2^2, \quad E = E_1 + E_2$$

γ_1 is bounded by a region A_1 , $|\gamma_1| \leq A_1 = \sqrt{2E_1(0)}$ & γ_2 oscillates within A_2 .

Their intersection is given by:



Where this rectangle is inscribed in an ellipse