1.1 Talk - Noether's Theorem and D'Alembert's Principle

Arnold, Mathematical Methods of Classical Mechanics, §20–21.

1.1.1 Noether's Theorem

Setting: M smooth manifold, $L : TM \to \mathbb{R}$ smooth, $h : M \to M$ smooth.

Definition 1.1 (Diffeomorphism)**.** $h : M \to M$ is a *diffeomorphism* if it is a smooth bijection with smooth inverse.

Definition 1.2 (One-parameter group)**.** A family of diffeomorphisms $h^s : M \to M$, $s \in \mathbb{R}$, depending smoothly on s, with

$$h^{s+t} = h^s \circ h^t, \qquad h^0 = \mathrm{id}.$$

Example 1.3 (Rotation). $h^s(x, y) = (x\cos s - y\sin s,\ x\sin s + y\cos s)$. Rotating by s then t = rotating by $s + t$. Velocity field:

$$\frac{d}{ds}\Big|_{s=0} h^s(x, y) = (-y,\ x).$$

A single rotation by 90° is not a family: no parameter s, no conservation law.

Definition 1.4 (Admits h)**.** (M, L) *admits the mapping* h if for any tangent vector $\mathbf{v} \in TM$,

$$L(h_*\mathbf{v}) = L(\mathbf{v}).$$

Here h_* acts on velocities by the Jacobian: $h_*(\mathbf{q}, \dot{\mathbf{q}}) = (h(\mathbf{q}),\ \frac{\partial h}{\partial \mathbf{q}}\dot{\mathbf{q}})$.

Example 1.5. $L = \frac{m}{2}(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2) - U(x_2, x_3)$ admits $x_1 \mapsto x_1 + s$, but generally not $x_2 \mapsto x_2 + s$. Pendulum: $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$ does *not* admit $\theta \mapsto \theta + s$, since $\cos(\theta + s) \neq \cos\theta$.

Theorem 1.6 (Noether)**.** *If (M, L) admits the one-parameter group of diffeomorphisms $h^s : M \to M$, $s \in \mathbb{R}$, then the lagrangian system corresponding to L has a first integral $I : TM \to \mathbb{R}$. In local coordinates $\mathbf{q}$ on M,*

$$\boxed{I(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \dot{\mathbf{q}}} \frac{dh^s(\mathbf{q})}{ds}\Big|_{s=0}}$$

(First integral: $\frac{d}{dt} I(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = 0$ along every solution.)

Proof. Take $M = \mathbb{R}^n$. Let $\mathbf{q} = \varphi(t)$ be a solution. Since h^s preserves L, $h^s \circ \varphi$ is also a solution. Let

$$\Phi(s, t) = h^s(\varphi(t)), \qquad \dot{} = \tfrac{d}{dt}, \quad {}' = \tfrac{d}{ds}.$$

1. L does not change in s:
$$0 = \frac{\partial L(\Phi, \dot{\Phi})}{\partial s} = \frac{\partial L}{\partial \mathbf{q}}\Phi' + \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\Phi}'. \tag{1}$$

2. Each $\Phi(s, \cdot)$ solves Lagrange's equation: $\dfrac{\partial L}{\partial \mathbf{q}} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\mathbf{q}}}$. Substitute into (1):
$$0 = \Big(\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}}\Big)\mathbf{q}' + \frac{\partial L}{\partial \dot{\mathbf{q}}}\Big(\frac{d}{dt}\mathbf{q}'\Big) = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\mathbf{q}}}\,\mathbf{q}'\Big) = \frac{dI}{dt}.$$

3. At $s = 0$, $\mathbf{q}' = \frac{d}{ds}h^s(\mathbf{q})\big|_{s=0}$, so this is I. □

1.1.2 Examples of Noether's Theorem

Recipe: (1) find h^s; (2) check L is preserved; (3) compute $\frac{d}{ds}h^s\big|_{s=0}$; (4) dot with $\partial L/\partial\dot{\mathbf{q}}$.

Example 1.7 (Momentum). $L = \sum m_i\frac{\dot{\mathbf{x}}_i^2}{2} - U(\mathbf{x})$, admitting $h^s : \mathbf{x}_i \mapsto \mathbf{x}_i + s\mathbf{e}_1$. Then $\frac{d}{ds}h^s\mathbf{x}_i = \mathbf{e}_1$ and

$$I = \sum \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\,\mathbf{e}_1 = \sum m_i\dot{x}_{i1} = \boxed{P_1}.$$

Example 1.8 (Angular momentum). h^s = rotation about $\mathbf{e}_1$ by angle s:

$$h^s\mathbf{x} = (x_1,\ x_2\cos s - x_3\sin s,\ x_2\sin s + x_3\cos s), \qquad \frac{d}{ds}\Big|_0 h^s\mathbf{x} = (0, -x_3, x_2) = [\mathbf{e}_1, \mathbf{x}].$$

Using $(\mathbf{a}, [\mathbf{b}, \mathbf{c}]) = (\mathbf{b}, [\mathbf{c}, \mathbf{a}])$:

$$I = \sum (m_i\dot{\mathbf{x}}_i, [\mathbf{e}_1, \mathbf{x}_i]) = \sum ([\mathbf{x}_i, m_i\dot{\mathbf{x}}_i], \mathbf{e}_1) = \boxed{M_1}.$$

Example 1.9 (Orbit). $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$ admits $\theta \mapsto \theta + s$, so $I = \partial L/\partial\dot{\theta} = \boxed{mr^2\dot{\theta}}$ (Kepler's second law).

Example 1.10 (Problem 4: energy). For $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, make time a coordinate: on $M \times \mathbb{R}$ with parameter τ ($' = d/d\tau$),

$$L_1(\mathbf{q}, t, \mathbf{q}', t') = L\Big(\mathbf{q}, \frac{\mathbf{q}'}{t'}, t\Big)\,t', \qquad \int L\,dt = \int L_1\,d\tau.$$

If L has no t, L_1 admits $h^s(\mathbf{q}, t) = (\mathbf{q}, t + s)$, velocity field $\partial/\partial t$. So

$$I_1 = \frac{\partial L_1}{\partial t'} = L + t'\frac{\partial L}{\partial \dot{\mathbf{q}}}\Big(-\frac{\mathbf{q}'}{t'^2}\Big) = L - \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} \quad\Longrightarrow\quad \boxed{E = \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} - L \text{ conserved}}.$$

1.1.3 D'Alembert's Principle

Point $\mathbf{x}$ of mass m must stay on a surface $M \subset \mathbb{R}^3$; $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - U(\mathbf{x})$. With $U = 0$, Newton gives a straight line, which leaves M: a new force is needed.

Definition 1.11 (Constraint force)**.** $\mathbf{R} = m\ddot{\mathbf{x}} + \dfrac{\partial U}{\partial \mathbf{x}}$. (Ideal constraint: $\mathbf{R} \perp M$, i.e. $(\mathbf{R}, \xi) = 0$ for all $\xi \in T_{\mathbf{x}}M$.)

Tangent plane of $M = \{f = 0\}$: $T_{\mathbf{x}}M = \{\xi : (\nabla f(\mathbf{x}), \xi) = 0\}$. Sphere $|\mathbf{x}|^2 = r^2$: $T_{\mathbf{x}}M = \{\xi \perp \mathbf{x}\}$.

Definition 1.12 (D'Alembert–Lagrange principle)**.** Tangent vectors $\xi \in T_{\mathbf{x}}M$ are *virtual variations*. For every virtual variation,

$$\boxed{\Big(m\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}},\ \xi\Big) = 0}$$

i.e. the work of the constraint force on any virtual variation is zero.

Definition 1.13 (Conditional extremal)**.** $\mathbf{x} : [t_0, t_1] \to M \subset \mathbb{R}^N$ is a *conditional extremal* of $\Phi = \int_{t_0}^{t_1}\big(\frac{\dot{\mathbf{x}}^2}{2} - U(\mathbf{x})\big)dt$ if $\delta\Phi = 0$ for variations by nearby curves in M with the same endpoints. Written $\delta_M\Phi = 0$; equivalent to Lagrange's equations on M.

Theorem 1.14. $\mathbf{x}$ *is a conditional extremal* ($\delta_M\Phi = 0$) *if and only if*

$$\Big(\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}},\ \xi\Big) = 0 \quad \forall\,\xi \in T_{\mathbf{x}}M.$$

Lemma 1.15. *If* $\mathbf{f}(t)$ *is continuous and* $\int_{t_0}^{t_1} \mathbf{f}(t)\,\xi(t)\,dt = 0$ *for every continuous* $\xi(t) \in T_{\mathbf{x}(t)}M$ *with* $\xi(t_0) = \xi(t_1) = 0$*, then* $\mathbf{f}(t) \perp M$ *at every* $\mathbf{x}(t)$*.*

Proof of the theorem. Vary $\mathbf{x}$ to $\mathbf{x} + \xi$, $\xi(t_0) = \xi(t_1) = 0$. Integrate by parts (boundary terms vanish):

$$\delta\Phi = \int_{t_0}^{t_1} \Big(\dot{\mathbf{x}}\dot{\xi} - \frac{\partial U}{\partial \mathbf{x}}\xi\Big)dt = -\int_{t_0}^{t_1} \Big(\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}}\Big)\xi\,dt.$$

($\mathbf{x} + \xi$ leaves M only to second order.) So $\delta_M\Phi = 0 \iff$ this integral vanishes for all tangent $\xi \iff$ (lemma, $\mathbf{f} = \ddot{\mathbf{x}} + \partial U/\partial \mathbf{x}$) D'Alembert's equation. □

1.1.4 Examples of D'Alembert's Principle

Example 1.16 (Pendulum). $\mathbf{x} = l(\sin\theta, -\cos\theta)$, $U = mgy$, $\xi = (\cos\theta, \sin\theta)$.

$$\ddot{\mathbf{x}} = l\ddot{\theta}(\cos\theta, \sin\theta) + l\dot{\theta}^2(-\sin\theta, \cos\theta),$$

$$(m\ddot{\mathbf{x}} + mg\mathbf{e}_y,\ \xi) = ml\ddot{\theta} + mg\sin\theta = 0 \implies \boxed{\ddot{\theta} = -\tfrac{g}{l}\sin\theta}.$$

The tension never appears.

Example 1.17 (Bead on a rotating rod, Arnold p. 95). Rod at angle α to the vertical, rotating with angular velocity ω; q = distance from O.

$$\mathbf{x} = q\,\mathbf{e}(t), \quad \mathbf{e}(t) = (\sin\alpha\cos\omega t,\ \sin\alpha\sin\omega t,\ \cos\alpha), \quad \xi = \mathbf{e}(t).$$

$(\mathbf{e}, \mathbf{e}) = 1$, $(\mathbf{e}, \dot{\mathbf{e}}) = 0$, $(\mathbf{e}, \ddot{\mathbf{e}}) = -|\dot{\mathbf{e}}|^2 = -\omega^2\sin^2\alpha$. From $\ddot{\mathbf{x}} = \ddot{q}\,\mathbf{e} + 2\dot{q}\,\dot{\mathbf{e}} + q\,\ddot{\mathbf{e}}$:

$$(\ddot{\mathbf{x}}, \mathbf{e}) = \ddot{q} - \omega^2\sin^2\alpha\, q = 0 \implies \boxed{\ddot{q} = \omega^2\sin^2\alpha\, q},$$

the same as Lagrange's equation from $L = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2\sin^2\alpha$.
$\mathbf{R} = m\ddot{\mathbf{x}} \perp$ rod, but $(\mathbf{R}, \dot{\mathbf{x}}) = 2m\omega^2\sin^2\alpha\, q\dot{q} \neq 0$: it does work on the actual path.

Example 1.18 (Free particle on a sphere). $|\mathbf{x}| = r$, $U = 0$, $\xi \perp \mathbf{x}$. Then $\ddot{\mathbf{x}} = \lambda\mathbf{x}$; differentiating $(\mathbf{x}, \mathbf{x}) = r^2$ twice gives $\lambda = -|\dot{\mathbf{x}}|^2/r^2$. Speed is constant and $\frac{d}{dt}[\mathbf{x}, \dot{\mathbf{x}}] = 0$, so the motion is a $\boxed{\text{great circle}}$.

1.1.5 Statics

Theorem 1.19. $\mathbf{x}_0 \in M$ *is an equilibrium position if and only if* $(\mathbf{f}(\mathbf{x}_0), \xi) = 0$ *for all* $\xi \in T_{\mathbf{x}_0}M$*, where* $\mathbf{f} = -\partial U/\partial \mathbf{x}$*.*

Definition 1.20 (Force of inertia)**.** $-m\ddot{\mathbf{x}}$ is called the *force of inertia.*

Theorem 1.21. *If the forces of inertia are added to the acting forces,* $\mathbf{x}$ *becomes an equilibrium position:* $(-m\ddot{\mathbf{x}} + \mathbf{f},\ \xi) = 0$*.*